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Study of QCD medium by sum rules

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Abstract : Though it has no analogue in condensed matter physics, the thermal QCD sum rules can, nevertheless, answer questions of condensed matter type about the QCD medium. The ingredients needed to write such sum rules, *viz*, the operator product expansion and the spectral representation at finite temperature, are reviewed in detail The sum rules are then actually written for the case of correlation function of two vector currents. Collecting information on the thermal average of the higher dimension operators from other sources, we evaluate these sum rules for the temperature dependent ρ-meson parameters. Possibility of extracting more information from the combined set of all sum rules from different correlation functions is also discussed

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1. Introduction

Ever since the suggestion by Nambu [1] that the physical vacuum of a quantum field theory for strongly interacting particles is analogous to the ground state of an interacting many-body system, condensed matter physics and particle physics enjoy a deep similarity in their basic ideas. Indeed, the idea of spontaneous symmetry breaking, observed in a ferromagnetic system, has been a basic ingredient in constructing relativistic quantum field theories, like the QCD theory, the electroweak theory and the grand unified theories. In more recent times, notions of condensed matter physics pervaded two more topics of particle physics, *viz*, that of the early universe and of the heavy ion collisions. Here an important problem to investigate is the restoration of symmetry at high temperature giving rise to a phase transition in such systems.

In a sense, the ideas of condensed matter physics are better tested in applications to particle physics. Take, for example, the phenomenon of phase transition. In condensed matter systems, it is based on the phenomenological Lagrangian, written by Landau for the sole purpose of discussing this particular phenomenon. On the other hand, phase transition in hadronic system, for example, is discussed on the basis of the QCD Lagrangian, which has found innumerable experimental support. Condensed matter physics has also been benefitted by feedbacks from particle physics investigations [2]. Sometimes the basic idea behind a condensed matter phenomenon has been pinned down by comparing with an analogous situation in particle physics. Clarifying the role of electromagnetic gauge symmetry in superconductivity is an example [3].

With such a close structural similarity between the theories of condensed matter and of particle physics, it is natural to expect similar calculational techniques for answering similar questions. One such method is the numerical calculation on the lattice, though working with the gauge and the fermi fields on the lattice is quite involved. It is ultimately going to be the quantitative method for these investigations. There is also the analytic method of effective field theories, which can be applied to both branches [2]. Being non-renormalisable, it brings in new constants at each higher order loop calculation. In particle physics, this method is called chiral perturbation theory [4], a low energy theory based on the pion degrees of freedom. It has been used to calculate the partition function and the resulting physical properties to three loops [5].

Of course, particle physics has also other aspects not shared by condensed matter physics. The most prominent differing aspects arise from the existence of increasing number of distinct quantum fields at higher and higher energies, which are assumed to be fundamental at present. In QCD theory the strong colour force among the field quanta give rise to bound states and resonances at low energy. The properties of such low energy states can be related to the vacuum structure of the QCD theory through the so-called QCD sum rules [6], which has no analogue in the condensed matter theory. When extended to finite temperature, the sum rules provide a method to investigate the properties of the QCD medium. It is this method of QCD sum rules that we are going to review in this article. More specifically, we shall try to explain at length the different elements involved in writing the sum rules and to show how their evaluation may address condensed matter questions on the QCD medium.

In Sec.2 we introduce the basic ideas of the sum rules and also discuss the earlier attempts to write them. The next two sections describe the two key elements needed in writing the QCD sum rules: Sec.3 describes a configuration space approach to find the Wilson coefficients in the operator product expansion of correlation functions and Sec.4 derives the spectral representation for the same at finite temperature. In Sec.5 we exemplify these discussions by actually writing down the thermal sum rules for the correlation function of two vector currents. The evaluation of these sum rules are carried out in Sec.6. Finally in Sec.7 we discuss the different aspects of these sum rules and a strategy for their evaluation.

2. About QCD sum rules

In the sum rule approach [6], one considers the correlation function of the product of two local operators, like the currents of the QCD theory. To begin with, consider the sum rule obtained by equating the dispersion relation for the vacuum matrix element of the operator product to its perturbative evaluation at sufficiently high euclidean momentum, where the QCD perturbation theory applies. At such high momentum, the continuum will contribute substantially to the dispersion integral along with the resonance(es). Such a sum rule, however, is not useful for predicting the parameters of the low energy resonances, as it is not possible to evaluate the continuum contribution reliably. On the other hand, if the momentum is low enough to emphasise the resonance contribution, the perturbative evaluation of the two point function will not be valid.

At this point Shifman, Vainshtein and Zakharov [6] suggested that as the momentum is lowered from high values, the perturbation theory first begins to fail not due to the growing magnitude of the coupling constant, but due to the existence of higher dimension operators. The contribution of these operators to the two point function is given by the Wilson operator product expansion [7]. The unit operator in the expansion stands for the perturbative result, while the higher dimension operators give rise to the so-called power corrections. It is the vacuum matrix elements of the higher dimension operators which characterise the low energy structure of the QCD theory. The idea of extending these vacuum sum rules to finite temperature by replacing the vacuum matrix element with the thermal average naturally suggests itself.

The original work establishing the thermal QCD sum rules is that of Bochkarev and Shaposhnikov [8]. Again one equates the spectral representation of the two point function to the corresponding operator product expansion. These thermal sum rules relate the temperature dependent resonance parameters to the thermal average of the higher dimension operators. Thus if these resonance parameters are obtained from field theory by calculating the self-energy functions at finite temperature, values for the thermal averages of the higher dimension operators can be extracted from the sum rules. These latter quantities hold nonperturbative information on the thermodynamic system described by the QCD theory. In particular, some of these thermal averages behave as order parameters, whose vanishing signals the existence of phase transition.

The work of ref [8] as well as the subsequent works [9] on thermal QCD sum rules were, however, flawed by the fact that there arise additional operators [10] in the Wilson expansion at finite temperature, which were not properly taken into account. The additional operators arise because of the breakdown of Lorentz invariance at finite temperature by the choice of the thermal rest frame, where matter is at rest at a definite temperature [11]. The residual $O(3)$ invariance naturally brings in additional operators. As we shall see, the expected behaviour of the thermal averages of these Lorentz non-invariant (new) operators is somewhat opposite to those of the Lorentz invariant (old) ones : While the old operators start with non-zero values at zero temperature and decrease in magnitude with the rise of temperature, the new ones, on the other hand, are zero at zero temperature but grow rapidly with temperature. The importance of including these new operators in the thermal sum rules, particularly at not too low a temperature, is thus clear.

It is convenient to write difference sum rules by subtracting the vacuum sum rules from their finite temperature counterparts. Such subtraction will reduce the importance of continuum contribution at higher energies, where it is almost temperature independent and also the contributions of higher dimension operators at low temperature. The problem is now to cope with the unknown thermal average of a rather large number of higher dimension operators. We shall take up this point in Sec.6 and 7.

Finally we mention some technical points. In the literature the vacuum sum rules are written for the time ordered product of operators, while for the thermal sum rules it is the retarded (or advanced) product which is preferred. But the use of the T-product, though a little complicated in writing down the spectral representation, has the advantage in perturbative calculations, where we can apply the conventional formalism. The other technical point to mention is that we shall employ the real time formulation of the thermal field theory [12], which requires in general not only the physical fields but also the accompanying 'ghost' fields. Since, however, we work to lowest order in perturbation expansion, ghost fields do not show up.

3. Operator product expansion

The use of the operator product expansion is at the heart of the QCD sum rules. It is the contribution of the higher dimension operators which is so essential for the success of the sum rules. Though basically perturbation theoretic, the calculation of the coefficients of the different operators is somewhat technical in nature. In their original work, Shifman *et. al.* [6] based this calculation on the Feynman diagrams, which was however very inconvenient. Later more elegant methods have been developed to calculate these coefficients [13]. In the following we describe a simple, configuration space method [14], [15] in the Fock-Schwinger gauge [16].

We find the operator product expansion of two quark bilinears of the QCD theory,

$$J_1(z) = \bar{\psi}(z)\,\Gamma_1\psi(z), \qquad J_2(z) = \bar{\psi}(z)\,\Gamma_2\psi(z),$$

where $\psi_a(z)$ is the quark field of a single flavour of mass m, the subscript a being its $SU(3)$ index. $\Gamma_{1,2}$ are any two of the Dirac matrices. Flavour changing currents can also be dealt with conveniently by introducing Pauli matrices in the flavour space.

Let us first consider *free* quark fields. The required operator product expansion in this case is given by the well-known Wick's theorem,

$$TJ_1(z)J_2(y) \tag{3.1}$$

$$= tr\,\Gamma_1 S^{(o)}(z,y)\,\Gamma_2 S^{(o)}(y,z) \tag{3.2}$$

$$-i(\Gamma_2 S^{(o)}(y,z)\,\Gamma_1)_{ab} : \bar{\psi}_a(y)\psi_b(z) : -i(\Gamma_1 S^{(o)}(z,y)\,\Gamma_2)_{ab} : \bar{\psi}_a(z)\psi_b(y) \tag{3.3}$$

$$+ : \bar{\psi}(z)\,\Gamma_1\psi(z)\,\bar{\psi}(y)\,\Gamma_2\psi(y) : \tag{3.4}$$

where tr indicates trace over both Dirac and colour matrices and : : indicated normal ordering. The c-number function $S^{(o)}_{ab}(z-y) = i\langle 0\,|\,T\psi_a(z)\,\bar{\psi}(y)\,|\,0\rangle$ satisfies

$$(-i\partial\!\!\!/_z + m)\,S^{(o)}(z-y) = \delta^4(z-y), \tag{3.5}$$

having the solution

$$S^{(o)}(z,y) = \frac{1}{4\pi^2}\left[\partial\!\!\!/\left(\frac{1}{(z-y)^2}\right) - \frac{im}{(z-y)^2} + \ldots\right]. \tag{3.6}$$

Expanding the quark field $\psi(z)$ around $z = y$ in (3.3), we can read off the Wilson coefficients multiplying different local operators.

Now switch on the interaction of quarks with gluons. If we ignore the quantum propagation of gluons, *i.e.*, treat them classically, the above Wick expansion still holds, but with $S^{(o)}(z,y)$ replaced by $S(z,y)$, satisfying,

$$(-iD\!\!\!\!/_z + m)\,S(z,y) = \delta^4(z-y), \quad D_\mu = \partial_\mu - iA_\mu\,, \quad A_\mu = gA^i_\mu\lambda^i/2, \tag{3.7}$$

where $A_\mu(z)$ are the (classical) gluon fields, g is their coupling to the quark fields and λ^i are the $SU(3)$ Gell-Mann matrices. It is possible to solve (3.7) in a short distance expansion for the singular terms of $S(z,y)$.

Short distance expansion of quark propagator

It is convenient to choose the Fock-Schwinger gauge $(z-y)^\mu A_\mu(z) = 0$ [16]. In this gauge the field strength $G_{\mu\nu}(z)$,

$$G_{\mu\nu}(z) = \partial_\mu A_\nu(z) - \partial_\nu A_\mu(z) - i\,[A_\mu(z), A_\nu(z)],$$

can be integrated to give

$$A_\mu(z) = \int_0^1 dt t z^\nu G_{\nu\mu}(tz).$$

In the following we put $y = 0$. Then the potential can be written as a series expansion in terms of the field strength and its covariant derivative at $z = 0$:

$$A_\mu(z) = \frac{1}{2} z^\alpha G_{\alpha\mu}(o) + \ldots,$$

The solution for $S(z, o)$ will now be automatically gauge covariant.

We convert (3.7) into a second order differential equation by setting

$$S(z,0) = (i\not{D} + m)\, E(z,0). \tag{3.8}$$

Then $E(z,0)$ satisfies

$$(\Box - P_\mu \partial_\mu - Q - m^2)\, E(z,0) = \delta^4(z), \tag{3.9}$$

where

$$P_\mu(z) = iz^\alpha G_{\alpha\mu}(o) + \ldots, \qquad Q(z) = \frac{i}{2}\gamma G \gamma + \ldots .$$

Here and below we use matrix notation, *e.g.*, $\gamma G \gamma = \gamma^\mu G_{\mu\nu}(o)\gamma^\nu$. The fundamental or the singular solution to (3.9) is now given by the Hadamard Ansatz for $E(z,0)$ [17],

$$E(z,0) = -\frac{i}{4\pi^2}\left(\frac{1}{z^2} + V(z)\, ln\,(4z^2\mu^2) + W(z)\right), \tag{3.10}$$

where $V(z)$ and $W(z)$ are analytic functions of z, regular in the neighbourhood of $z = 0$. The piece $W(z)$ is added to ensure that $E(z,0)$, by itself, satisfies eq. (3.9). The factor $4\mu^2$ in the argument of the logarithm represents a mass scale (not related to the quark mass), arbitrary at this stage. Inserting this Ansatz in (3.9), the elimination of singular terms like $1/z^2$ and lnz^2 on the left hand side requires that V and W satisfy

$$4z^\mu \partial_\mu V + 4V - Q = z^2 h(z), \tag{3.11}$$

$$(\Box - P^\mu \partial_\mu - Q)V = 0, \tag{3.12}$$

$$(\Box - P^\mu \partial_\mu - Q)W = -h(z). \tag{3.13}$$

Where $h(z)$ is an unknown function, regular in z, to be determined by the two equations for V. Thus the original problem of finding E reduces to solving the above three partial differential equations for V and W, which can be readily solved in power series in z^μ.

[1]For the time ordered propagator z^2 stands for $z^2 - i\epsilon$

The solution for $S(z, 0)$ is worked out in ref. [15] up to gauge field strengths of dimension six. Here we reproduce this solution up to dimension four,

$$S(z,0)=\frac{1}{4\pi^2}\left[\partial\!\!\!/\left(\frac{1}{z^2}\right)-\frac{im}{z^2}-\frac{i}{4z^2}\gamma^\mu z\!\!\!/\gamma^\nu G_{\mu\nu}+\frac{1}{24z^2}z\!\!\!/(zG\gamma)^2\right.$$

$$+\left(-\frac{1}{24}zGG\gamma+\frac{1}{16}(zG\gamma)(\gamma G\gamma)+\frac{z\!\!\!/}{32}\left\{\frac{1}{3}\gamma GG\gamma-\frac{1}{2}(\gamma G\gamma)^2\right\}\right)ln(4z^2\mu^2)$$

$$\left.-\frac{z\!\!\!/}{64}\left\{\frac{1}{3}\gamma GG\gamma-\frac{1}{2}(\gamma G\gamma)^2\right\}\right] \tag{3.14}$$

Again the last term, which is regular, ensures that $S(z, 0)$, by itself, satisfies (3.7) up to dimension four. Here we have omitted terms involving the gauge field covariant derivatives, as they do not survive the trace over colour in the correlation functions. In this case $S(z, 0)$ satisfies, $S(0, z) = S(-z, 0)$.

Once we have found the c-number coefficients multiplying the classical gauge field strengths in $S(z, 0)$, the latter can be considered as gauge field operators of QCD theory. Eq (3.14) thus constitute the short distance expansion of the two point function of the quark fields. Of course, higher order corrections to the coefficients due to gluon propagation is beyond the present method.

Wilson coefficients

We remind the reader that the Wick expansion for the *interacting* case is given by the terms (3.2-4) with $S^{(0)}$ replaced by S and we have set $y = 0$. The first term (3.2) gives the coefficients of the gluon operators. The second and third terms in (3.3) after expanding $\psi(z)$ in the bilocal operators as,

$$\psi(z)=\psi(0)+z^\mu\ D_\mu\psi(0)+\ldots,$$

gives the coefficients of all quark operators,

$$-\frac{1}{2\pi^2(z^2)^2}\left[\left\{i(\Gamma_2 z\!\!\!/\Gamma_1-\Gamma_1 z\!\!\!/\Gamma_2)_{ab}+mz^2(\Gamma_1\Gamma_2+\Gamma_2\Gamma_1)_{ab}/2\right\}:\overline{\psi}_a\psi_b\right.$$

$$\left.+(\Gamma_2 z\!\!\!/\Gamma_1+\Gamma_1 z\!\!\!/\Gamma_2)_{ab}z^\mu:\overline{\psi}_a iD_\mu\psi_b:\right] \tag{3.15}$$

The fourth term in (3.4) is regular and of no interest to us. Thus the terms (3.2-3) constitute the most general operator product expansion of two quark bilinears, giving the coefficients of all the gluon and quark operators up to dimension four, with various tensor and spinor structures.

Clearly at the level of operators there is no reference to temperature. It is in taking the matrix elements of these operators that the difference between the vacuum state and the thermal state shows up. In taking the vacuum expectation value, the different tensor and the spinor operators are projected on to their corresponding Lorentz scalars. The situation is different for the thermal expectation value, which for any operator O is defined as

$$\langle O\rangle = Tr[e^{-\beta H}\ O]/Tr[e^{-\beta H}],$$

where H is the QCD Hamiltonian, β is the inverse temperature T and the Trace is over any complete set of states. Here the choice of the thermal rest frame, where the temperature is defined, breaks the Lorentz invariance and we have to include $O(3)$ invariant operators as well.

It is, however, convenient for book-keeping with indices to restore Lorentz invariance formally in the thermal field theory by introducing the four-vector u^μ, the velocity of the heat bath. [$u^2 = 1$ and $u^\mu = (1, 0, 0, 0)$ in the rest frame of the heat bath.] Thus while at zero temperature scalar operators are formed from tensors by contracting its indices among themselves, at finite temperature additional scalars can be formed by contracting its indices with u^μ [18].

Consider first the piece (3.2) with gluon operators. The Lorentz covariance at finite temperature now allows us to write

$$\left\langle tr^c G_{\alpha\beta} G_{\lambda\sigma} \right\rangle = (g_{\alpha\lambda} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\lambda})\, A$$
$$- (u_\alpha u_\lambda g_{\beta\sigma} - u_\alpha u_\sigma g_{\beta\lambda} - u_\beta u_\lambda g_{\alpha\sigma} + u_\beta u_\sigma g_{\alpha\lambda})\, B \tag{3.16}$$

By contracting indices on both sides, we get

$$A = \frac{1}{24} \left\langle G^a_{\alpha\beta}\ G^{a\alpha\beta} \right\rangle + \frac{1}{6} \left\langle u^\alpha \Theta^g_{\alpha\beta} u^\beta \right\rangle, \tag{3.17}$$

$$B = \frac{1}{3} \left\langle u^\alpha \Theta^g_{\alpha\beta} u^\beta \right\rangle, \tag{3.18}$$

$\Theta^g_{\alpha\beta}$ being the traceless, gluonic part of the stress-tensor of the QCD theory,

$$\Theta^g_{\alpha\beta} = - G^a_{\alpha\lambda} G^{\lambda a}_{\beta} + \frac{1}{4} g_{\alpha\beta} G^a_{\lambda\sigma} G^{\lambda\sigma a} \tag{3.19}$$

Since we are interested in the coefficients of operators of dimension four only, we may already use (3.16) in the expression for $S(z, 0)$, which then simplifies to

$$S(z, 0) = \frac{1}{4\pi^2} \left(\partial\!\!\!/ \left(\frac{1}{z^2} \right) - \frac{i}{4z^2}\, \gamma^\mu t \gamma^\nu G_{\mu\nu} + \frac{B}{48} (a t + b u\!\!\!/) \right), \tag{3.20}$$

where

$$a = 1 - \frac{4}{z^2} (u \cdot z)^2 - 2 ln\, (4z^2 \mu^2), \quad b = 8u\,.\, z ln\, (4z^2 \mu^2)\,.$$

Insert this expression for $S(z, 0)$ in (3.2) and use (3.16) again for the dimension four operators generated in the product. Then the coefficients of the operators in A and B are obtained as traces over γ-matrices.

Next we turn to the piece (3.3) with quark and gluon operators. As with the gluon operators, we project the quark operators on the corresponding scalars. At finite temperature the most general decomposition of the two operators in (3.3) in spinor space, conserving parity, are given by

$$\left\langle \bar\psi_a \psi_b \right\rangle = \frac{\delta_{ab}}{4} \left\langle \bar\psi\psi \right\rangle + \frac{(u\!\!\!/)_{ba}}{4} \left\langle \bar\psi u\!\!\!/ \psi \right\rangle \tag{3.21}$$

$$\left\langle \bar\psi_a i D_\mu \psi_b \right\rangle = (\gamma_\mu)_{ba} U + u_\mu (u\!\!\!/)_{ba} V + u_\mu \delta_{ba} Y + (u\!\!\!/ \gamma_\mu)_{ba} Z \tag{3.22}$$

where

$$U = \frac{m}{16}\langle\bar{\psi}\psi\rangle - \frac{1}{12}\langle u^\mu \Theta^f_{\mu\nu} u^\nu\rangle, \quad V = \frac{1}{3}\langle u^\mu \Theta^f_{\mu\nu} u^\nu\rangle, \tag{3.23}$$

$$Y = \frac{1}{6}\langle\bar{\psi} i u\cdot D\psi\rangle + \frac{m}{12}\langle\bar{\psi}\not{u}\psi\rangle, \quad Z = \frac{1}{12}\langle\bar{\psi} i u\cdot D\psi\rangle - \frac{m}{12}\langle\bar{\psi}\not{u}\psi\rangle \tag{3.24}$$

and $\Theta^f_{\mu\nu}$ is the traceless, fermionic part of the stress tensor,

$$\Theta^f_{\mu\nu} = \bar{\psi}\gamma_\mu iD_\nu\psi - \frac{1}{4} g_{\mu\nu} m\bar{\psi}\psi .$$

Now $\bar{\psi} iu\cdot D\psi$ can be reduced to $m\bar{\psi}\not{u}\psi$ and a total derivative. Further $\langle\bar{\psi}\not{u}\psi\rangle = 0$, for zero chemical potential and we shall not consider it any further. We are thus left with only $\langle\bar{\psi}\psi\rangle$ in (3.21) and U and V in (3.22). Inserting these decompositions in (3.3), we can get the coefficients of these operators.

Thus the calculation of the Wilson coefficients of different operators in the expansion of the product of two quark bilinears reduces to evaluating the traces over γ-matrices. These coefficients have to be Fourier transformed to momentum space for use in the QCD sum rules. In Sec. 6 we shall write explicitly the results for the correlation function of two vector currents

4. Spectral Representation

The conventional (zero temperature) sum rules are obtained by considering the vacuum expectation value of the two point function of quark bilinears. Using Lorentz invariance and the fact that the intermediate energies are all positive, one gets the spectral representation of the Källen-Lehmann type.

To get sum rules at finite temperature, we consider the thermal average of the correlation function

$$T_{12}(q) = i\int d^4x e^{iqx}\langle T J_1(x) J_2(0)\rangle, \tag{4.1}$$

We obtain the spectral representation for the correlation function in q_0 at fixed $|\boldsymbol{q}|$. First, evaluate the trace over a complete set of eigenstates of four-momentum, when it becomes a sum over forward amplitudes weighted by the corresponding Boltzmann factors,

$$T_{12}(q) = Z^{-1}\sum_m e^{-\beta E_m} i\int d^4x e^{iq\cdot x}\langle m\,|\,T J_1(x) J_2(0)|\,m\rangle, \tag{4.2}$$

Then insert the same set of complete states between the quark bilinears,

$$T_{12}(q) = Z^{-1}\sum_{m,n} ie^{-\beta E_m}\int d^4x e^{iq\cdot x}\{\theta(x^0)\langle m\,|\,J_1(x)|\,n\rangle\langle n\,|\,J_2(0)|\,m\rangle$$

$$+\,\theta(-x^0)\langle m\,|\,J_2(0)|\,n\rangle\langle n\,|\,J_1(x)|\,m\rangle\}, \tag{4.3}$$

to extract its x-dependence, using the formula, $O(x) = e^{iPx}O(o)\,e^{iPx}$. Integration over space gives rise to a three-dimensional delta-function, while integration over time gives the energy denominator. Finally we insert a delta-function in q_0 to write,

$$T_{12}(q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq_0'\left(\frac{M_{12}(q_0',|\boldsymbol{q}|)}{q_0'-q_0-i\varepsilon} - \frac{M_{21}(q_0',|\boldsymbol{q}|)}{q_0'-q_0+i\varepsilon}\right), \tag{4.4}$$

where

$$M_{12}(q) = Z^{-1} \sum e^{-\beta E_m} \delta^4 (q + p_m - p_n) \langle m | J_1 (0) | n \rangle \langle n | J_2 (0) | m \rangle \tag{4.5}$$

The double sum over states may be converted back to the form,

$$M_{12}(q) = \int d^4 x e^{iq \cdot x} \langle J_1(x) J_2(0) \rangle . \tag{4.6}$$

Because of the symmetry between m and n in the double sum representation, we have

$$M_{21}(q_0, |\boldsymbol{q}|) = e^{-\beta q_0} M_{12} (q_0, |\boldsymbol{q}|) . \tag{4.7}$$

The opposite sign of $i\varepsilon$ in the two terms in eq. (4.4) is typical of T-products. As a result the imaginary part of $\mathcal{T}_{12}$ is given by the sum, $\frac{1}{2}(M_{12} + M_{21})$, while the principal value integral contains the difference, $\frac{1}{2}(M_{12} + M_{21})$. But the two are related by (4.7),

$$M_{12} - M_{21} = (M_{12} + M_{21}) \tanh (\beta q_0 / 2). \tag{4.8}$$

We thus get the Landau representation for the T product at finite temperature [19],

$$\mathcal{T}_{12}(q_0, |\boldsymbol{q}|) = i Im \mathcal{T}_{12}(q_0, |\boldsymbol{q}|) + P \int_{-\infty}^{+\infty} dq_0' \frac{N_{12}(q_0', |\boldsymbol{q}|)}{q_0' - q_0} , \tag{4.9}$$

where, for brevity, we write

$$N_{12}(q) = \pi^{-1} \, Im \mathcal{T}_{12}(q) \tanh (\beta q_0 / 2) \tag{4.10}$$

It may actually require subtractions, but it does not affect the Borel transformed sum rules we shall write below.

5. Sum rules for vector current

Having explained in general terms the key elements involved in writing down the sum rules for the correlation function of any two quark bilinears, we now apply them to the important case of the vector current [20]. Consider the thermal average of the time ordered (T) product of two currents,

$$\mathcal{T}_{\mu\nu}^{ab} (q) = i \int d^4 x e^{iq \cdot x} \left\langle T(V_\mu^a (x) \, V_\nu^b (0)) \right\rangle , \tag{5.1}$$

where $V_\mu^a(x)$ is the vector current (in the ρ meson channel),

$$V_\mu^a (x) = \bar{q}(x) \gamma_\mu (\tau^a / 2) q(x), \tag{5.2}$$

$q(x)$ being the field of the u and d quark doublet and τ^a the Pauli matrices.

Kinematics

As already mentioned, Lorentz invariance can be enforced at finite temperature by bringing in the velocity four-vector u^μ. The time and the space components of q^μ are then raised to the Lorentz scalars, $\omega = u \cdot q$ and $\bar{q} = \sqrt{\omega^2 - q^2}$. In such a Lorentz invariant framework, there are two independent conserved kinematic covariants into which $\tau^{ab}_{\mu\nu}(q)$ can be decomposed [21]. We choose them as

$$P_{\mu\nu} = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} - \frac{q^2}{\bar{q}^2}\tilde{u}_\mu \tilde{u}_\nu, \qquad Q_{\mu\nu} = \frac{q^4}{\bar{q}^2}\tilde{u}_\mu \tilde{u}_\nu,$$

where $\tilde{u}_\mu = u_\mu - \omega\, q_\mu / q^2$. The invariant decomposition of $\tau^{ab}_{\mu\nu}$ is then given by

$$\tau^{ab}_{\mu\nu}(q) = \delta^{ab}(Q_{\mu\nu}T_l + P_{\mu\nu}T_t), \tag{5.3}$$

where we take the invariant amplitudes $T_{l,t}$ as functions of q^2 and ω.

Notice that the kinematic covariants $P_{\mu\nu}$ and $Q_{\mu\nu}$ are free from singularities at $q^2 = 0$ (and also at $\bar{q}^2 = 0$). This is convenient at finite temperature as there are dynamical singularities extending up to $q^2 = 0$. With this choice of kinematic covariants, the dynamical singularities reside only in the invariant amplitudes.

To extract the invariant amplitudes from $\tau^{ab}_{\mu\nu}$, it is convenient to form the scalars $\Pi_{1,2}$,

$$\delta^{ab}\Pi_1(q) = T^{\mu ab}_{\mu}(q), \quad \delta^{ab}\Pi_2(q) = u^\mu T^{ab}_{\mu\nu}(q)\, u^\nu.$$

Then the invariant amplitudes are given by

$$T_l = \frac{1}{\bar{q}^2}\Pi_2, \quad T_t = -\frac{1}{2}\left(\Pi_1 + \frac{q^2}{\bar{q}^2}\Pi_2\right). \tag{5.4}$$

In the limit $\boldsymbol{q} = 0$, the amplitudes T_l and T_t are related. To see this we write the spatial components of $\tau_{\mu\nu}$ as

$$T^{ab}_{ij}(q) = \delta^{ab}\,[\delta_{ij}T_t - \hat{q}_i\hat{q}_j(T_t - q_0^2 T_l)].$$

where $\hat{q}_i$ is the ith component of the unit vector along $\boldsymbol{q}$. As $|\boldsymbol{q}| \to 0$, there cannot be any dependence on $|\boldsymbol{q}|$, getting

$$T_t(q_0, |\boldsymbol{q}| = 0) = q_0^2\; T_l(q_0, |\boldsymbol{q}| = 0). \tag{5.5}$$

Spectral representation

Following the derivation in Sec.4, the Landau representation for the T product of (5.1) is given by the analogue of (4.9),

$$T^{ab}_{\mu\nu}(q_0, |\boldsymbol{q}|) = i\, Im\, T^{ab}_{\mu\nu}(q_0, |\boldsymbol{q}|) + P\int_{-\infty}^{+\infty} dq_0' \, \frac{N_{\mu\nu}(q_0', |\boldsymbol{q}|)}{q_0' - q_0}, \tag{5.6}$$

where

$$N^{ab}_{\mu\nu}(q) \equiv \delta^{ab} N_{\mu\nu}(q) = \pi^{-1}\; Im\, T^{ab}_{\mu\nu}(q) \tanh(\beta q_0/2) \tag{5.7}$$

Expressing $q_\mu' = q_\mu + (q_0' - q_0)\, u_\mu$, we recover the representation for the invariant amplitudes $T_{l,t}$. Further, using the symmetry properties, $Im T_{l,t}(-q_\mu) = Im T_{l,t}(q_\mu)$ and going over to imaginary values of q_0 ($q_0^2 = -Q_0^2$, $Q_0^2 > 0$), for which $N_{\mu\nu}$ vanishes, these become,

$$T_{l,t}(q_0^2, |\boldsymbol{q}|) = \int_0^\infty dq_0'^2 \frac{N_{l,t}(q_0', |\boldsymbol{q}|)}{q_0'^2 + Q_0^2}. \tag{5.8}$$

Again the question of subtraction is not relevant for us.

Operator product expansion

With $\Gamma_1 = \gamma_\mu$, $\Gamma_2 = \gamma_\nu$, we evaluate the different traces over γ-matrices occurring in Sec.3 and Fourier transform the result to get the contributions to the invariant amplitudes $T_{l,t}$ for large euclidean momenta ($Q^2 = -q^2$) as,

$$T_l = -\frac{1}{8\pi^2} ln\left(\frac{Q^2}{\mu^2}\right) + \frac{1}{Q^4}\left[\hat{m}\langle\bar{q}q\rangle + \frac{\langle G^2\rangle}{24} + \frac{4}{3}\langle\Theta^f\rangle + \frac{g^2}{18\pi^2}\left\{ln\left(\frac{Q^2}{\mu^2}\right) + 2\gamma - \frac{9}{4} - 2\frac{\bar{q}^2}{q^2}\right\}\langle\Theta^g\rangle\right], \tag{5.9}$$

$$T_t = \frac{Q^2}{8\pi^2} ln\left(\frac{Q^2}{\mu^2}\right) - \frac{1}{Q^2}\left[\hat{m}\langle\bar{q}q\rangle + \frac{\langle G^2\rangle}{24} + \frac{4}{3}\left(1 + 2\frac{\bar{q}^2}{q^2}\right)\langle\Theta^f\rangle + \frac{g^2}{18\pi^2}\left\{\left(1 + 2\frac{\bar{q}^2}{q^2}\right)\left(ln\left(\frac{Q^2}{\mu^2}\right) + 2\gamma\right) - \frac{9}{4} - 7\frac{\bar{q}^2}{q^2}\right\}\langle\Theta^g\rangle\right]. \tag{5.10}$$

where we have included the contribution of the unit operator, the perturbative result. Here γ is the Euler constant. μ ($\simeq$ 1 GeV) is the scale at which all renormalisations are carried out. $\hat{m}$ is the degenerate mass of the u and the d quarks and $<\bar{q}q> = <\bar{u}u> = <\bar{d}d>$. The operator G^2 is quadratic in the gauge field strength $G^a_{\mu\nu}$, $(a = 1, \ldots, 8)$. $G^2 = (\alpha_s / \pi) G^a_{\mu\nu} G^{a\mu\nu}$, with $\alpha_s = g^2 / 4\pi$. Along with these two operators already contributing to zero temperature, we now have the linear combinations of two new ones, $\Theta^{f,g} \equiv u^\mu \Theta^{f,g}_{\mu\nu} u^\nu$.

The large q^2 behaviour of the Wilson coefficients can be improved by the renormalisation group equation. It effectively shifts the renormalisation scale of these coefficients from μ to $\sqrt{Q^2}$. The operators $g^2 G_{\mu\nu} G^{\mu\nu}$ and $m\bar{\psi}\psi$ and hence their coefficients are invariant under this change of scale. But it mixes the operators $\Theta^{f,g}_{\mu\nu}$ and $\Theta^g_{\mu\nu}$, a situation familiar from the analysis of deep inelasic scattering [22]. Let us write the contributions of Θ^f and Θ^g to any of $T_{l,t}$, for short, as

$$C_f n_f \langle\Theta^f\rangle|_\mu + C_g \langle\Theta^g\rangle|_\mu$$

where the coefficients $C_{f,g}| \equiv C_{f,g}(ln(Q^2/\mu^2), g(\mu^2), \omega^2/q^2)]$ can be read off from (5.9-10) and n_f is the number of flavours. The coefficients satisfy the coupled renormalisation group equations with an anomalous dimension matrix [23]. Diagonalising this matrix, we see that the combination of the coefficients ($n_f C_f + \frac{16}{3} C_g$), corresponding to the total energy-momentum tensor operator $\Theta \equiv (n_f \Theta^f + \Theta^g)$, has zero anomalous dimension, while the combination

$(C_f - C_g)$, corresponding to the operator $n_f\,(\frac{16}{3}\Theta^f - \Theta^g)$ has anomalous dimension, $-\frac{g^2}{(4\pi)^2}d$, with $d = \frac{4}{3}(\frac{16}{3} + n_f)$. Thus

$$C_f n_f \langle \Theta^f \rangle|_\mu + C_g \langle \Theta^g \rangle|_\mu$$

$$= \frac{1}{\frac{16}{3} + n_f}\left\{\left(n_f \bar{C}_f + \frac{16}{3}\bar{C}_g\right)\langle\Theta\rangle + a n_f \left(\bar{C}_f - \bar{C}_g\right)\left\langle \frac{16}{3}\Theta^f - \Theta^g \right\rangle|_\mu\right\} \tag{5.11}$$

where $\bar{C}_{f,g} = \bar{C}_{f,g}(1, g^2(Q^2))$ and $a = \left\{(g^2(\mu^2)/g^2(Q^2)\right\}^{-d/2b}$, $b = 11 - 2n_f/3$. Thus ignoring g^2-corrections (which for C_f is not calculable anyway by the present method), the improved contribution to T_l is,

$$T_l = -\frac{1}{8\pi^2} ln\left(\frac{Q^2}{\mu^2}\right) + \frac{1}{Q^4}\left(\hat{m}\langle\bar{q}q\rangle + \frac{\langle G^2 \rangle}{24} + \frac{4}{16 + 3n_f}\right.$$

$$\left.\left\{\langle\Theta\rangle + a(Q^2)\langle 16\Theta^f/3 - \Theta^g\rangle\right\}\right). \tag{5.12}$$

T_t is also given by the same expression, except for an overall factor of $-Q^2$ and a factor of $(1 - 2\bar{q}^2/Q^2)$ multiplying the term with Θ's.

We wish to emphasise the result of mixing of the operators Θ^f and Θ^g under the renormalisation group. The Wilson coefficients for Θ^f and Θ^g were originally, to leading order, of zeroth and first order in α_s, arising from Born and one-loop graphs respectively. However, due to operator mixing, the coefficients are drastically changed in that both Θ and $(16\Theta^f/3\ \Theta^g)$ have coefficients with leading terms of zeroth order in α_s. In (5.12) we retain only these leading terms.

Sum rules

We now equate the spectral representation and the result of operator product expansion for the amplitudes T_l and T_t at sufficiently high Q_0^2. Taking Borel transform we arrive at the thermal QCD sum rules [8] For T_l we get

$$\frac{1}{M^2}\int_0^\infty dq_0^2 e^{-q_0^2/M^2} N_l(q_0, |\boldsymbol{q}|) = e^{-|\boldsymbol{q}|^2/M^2}\left(\frac{1}{8\pi^2} + \frac{\langle O \rangle}{M^4}\right), \tag{5.13}$$

where $\langle O \rangle$ is the non-perturbative contribution of higher dimension operators,

$$\langle O \rangle = \hat{m}\langle\bar{q}q\rangle + \frac{\langle G^2 \rangle}{24} + \frac{4}{16 + 3n_f}\left\{\langle\Theta\rangle + a(M^2)\langle 16\Theta/3 - \Theta\rangle\right\}. \tag{5.14}$$

and a similar one for T_t. Each is a two parameter sum rule, dependent not only on T but also on $|\boldsymbol{q}|$.

In the thermal rest frame, the thermal average of the new operators are energy densities, which increase rapidly with temperature. Earlier works on thermal QCD sum rules were flawed, as these contributions were not properly included.

Absorptive parts

We work below the critical temperature, where hadrons constitute the physical spectrum. As with the vacuum sum rules, the dominant contribution to the spectral function is given by the ρ-meson. We also calculate the contribution of the non-resonant $\pi\pi$-continuum. The question of other significant contributions will be discussed in Sec. 6.

The coupling of the vector current to the ρ-meson is given by

$$< 0 \mid V_\mu^a \mid \rho^b > = \delta^{ab} F_\rho m_\rho \varepsilon_\mu , \tag{5.15}$$

where ε_μ is the polarisation vector of ρ of mass m_ρ. The experimental value of F_ρ as measured by the electronic decay rate of ρ^0 [24] is $F_\rho = 153.3$ MeV.

A simple way to calculate the absorptive part of the ρ-pole diagram is to note the field-current identity of $V_\mu^a(x)$ with the rho-meson field ρ_μ^a,

$$V_\mu^a(x) = m_\rho F_\rho \rho_\mu^a(x) , \tag{5.16}$$

which reproduces (5.15). Then the ρ-meson contribution is given essentially by its thermal propagator,

$$\begin{aligned} T_{\mu\nu}^{ab}(q) &= i m_\rho^2 F_\rho^2 \int d^4x e^{iqx} \left\langle \rho_\mu^a(x) \rho_\nu^b(o) \right\rangle \\ &= i\delta^{ab} m_\rho^2 F_\rho^2 \left(-\eta_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2} \right) \Delta_{11}^\rho(q) , \end{aligned} \tag{5.17}$$

where $\Delta_{11}^\rho(q)$ is the 11-component of a scalar field propagator with mass m_ρ,

$$\Delta_{11}^\rho(q) = \frac{i}{q^2 - m_\rho^2 + i\varepsilon} + 2\pi n(\omega_q)\, \delta(q^2 - m_\rho^2) , \tag{5.18}$$

and $n(\omega_q)$ is the Bose distribution function, $n(\omega_q) = (e^{\beta\omega_q} - 1)^{-1}$, $\omega_q = \sqrt{\vec{q}^{\,2} + m_\rho^2}$. We then get

$$\begin{pmatrix} N_l \\ N_t \end{pmatrix} = \begin{pmatrix} 1 \\ m_\rho^2 \end{pmatrix} F_\rho^2 \delta \{ q_0^2 - (|\boldsymbol{q}|^2 + m_\rho^2) \} \tag{5.19}$$

Loop corrections at finite temperature will make m_ρ and F_ρ temperature dependent, modifying them, in general, differently in the longitudinal and transverse amplitudes. These modifications may be obtained by calculating the appropriate loop graphs. Here we shall find them by evaluating our sum rules.

The non-resonant $\pi\pi$-contribution to the amplitudes describes the interaction of the current with the particles in the medium, which are predominantly pions. Chiral perturbation theory [4] gives the contribution of the pion field $\phi^a(x)$ to the vector current, which, to lowest order, is

$$V_\mu^a(x) = \varepsilon^{abc} \phi^b(x)\, \partial_\mu \phi^c(x)$$

Then the pions contribute to the correlation function as

$$T^{ab}_{\mu\nu}(q) = i\delta^{ab} \int \frac{d^4k}{(2\pi)^4} (2k-q)_\mu (2k-q)_\nu \Delta^\pi_{11}(k)\, \Delta^\pi_{11}(k-q), \tag{5.20}$$

where Δ^π_{11} is again the 11-component of the scalar propagator (5.18) but with mass m_π.

It is now straightforward to obtain the imaginary part of (5.20) either directly or by the cutting rules at finite temperature [25]. The results are

$$\begin{pmatrix} N_l^+ \\ N_t^+ \end{pmatrix} = \frac{v^3}{32\pi^2} \begin{pmatrix} 1 \\ q^2 \end{pmatrix} + \begin{pmatrix} \overline{N}_l^+ \\ \overline{N}_t^+ \end{pmatrix}, \quad \text{for } q^2 > 4m_\pi^2 \tag{5.21}$$

with

$$\begin{pmatrix} \overline{N}_l^+ \\ \overline{N}_t^+ \end{pmatrix} = \frac{1}{32\pi^2} \int_{-v}^{v} dx \begin{pmatrix} 2x^2 \\ q^2(v^2 - x^2) \end{pmatrix} n((|\boldsymbol{q}|x + q_0)/2) \tag{5.22}$$

and

$$\begin{pmatrix} N_l^- \\ N_t^- \end{pmatrix} = \frac{1}{64\pi^2} \int_{v}^{\infty} dx \begin{pmatrix} 2x^2 \\ q^2(v^2 - x^2) \end{pmatrix}$$

$$\{n((|\boldsymbol{q}|x - q_0)/2) - n((|\boldsymbol{q}|x + q_0)/2)\}, \quad \text{for } \quad q^2 \leq 0 \tag{5.23}$$

where $v = \sqrt{1 - 4m_\pi^2/q^2}$. The superscripts ($\pm$) on N denote time-like and space-like q_μ respectively, where they are non-vanishing. The first term on the right of (5.21) is the zero temperature contribution of the $\pi\pi$ state. Evaluated here in a non-covariant way it, of course, agrees with the covariant evaluation of the Feynman amplitude (5.20) with $\Delta^\pi_{11}(k)$ replaced by the vacuum propagator [4].

Explicit sum rules

Let us now write explicitly the sum rule (5.13) for T_l. Saturating N_l with the above contributions it becomes,

$$F_\rho^2(T) e^{-m_\rho^2(T)/M^2} + \frac{1}{48\pi^2} \int_{4m_\pi^2}^{\infty} dq_0^2 e^{-q_0^2/M^2} v^3 (q_0^2 + |\boldsymbol{q}|^2)$$

$$+ e^{|\boldsymbol{q}|^2/M^2} \left(\int_{4m_\pi^2 + |\boldsymbol{q}|^2}^{\infty} dq_0^2 e^{-q_0^2/M^2}\, \overline{N}_l^+(q_0, |\boldsymbol{q}|) + \int_0^{|\boldsymbol{q}|^2} dq_0^2 e^{-q_0^2/M^2} N_l^-(q_0, |\boldsymbol{q}|) \right)$$

$$= \frac{M^2}{8\pi^2} + \frac{\langle O \rangle}{M^2} \tag{5.24}$$

As the temperature goes to zero, the two terms in bracket go to zero and the thermal average of the operators on the right become the vacuum expectation values, recovering the familiar vacuum sum rule. The integral on the left is the non-resonant 2π contribution and is small compared to the resonance contribution.

As $|q|\to 0$, the sum rule (5.24) simplifies considerably. The limit for the second integral in bracket is given in ref. [8]. The sum rules for T_l and T_t then become,

$$F_\rho^2(T)e^{-m_\rho^2(T)/M^2}+I_0(M^2)+I_T(M^2)=\frac{M^2}{8\pi^2}+\frac{\langle O\rangle}{M^2}, \tag{5.25}$$

and

$$M_\rho^2(T)\,F_\rho^2(T)e^{-m_\rho^2(T)/M^2}+J_0(M^2)+J_T(M^2)=\frac{M^4}{8\pi^2}-\langle O\rangle, \tag{5.26}$$

where

$$I_0(M^2)=\frac{1}{48\pi^2}\int_{4m_\pi^2}^{\infty}dse^{-s/M^2}v^3,$$

$$J_0(M^2)=\frac{1}{48\pi^2}\int_{4m_\pi^2}^{\infty}dsse^{-s/M^2}v^3,$$

$$I_T(M^2)=\frac{1}{24\pi^2}\int_{4m_\pi^2}^{\infty}ds\,\{e^{-s/M^2}v^3+v(3-v^2)/2\}\,n(\sqrt{s}/2),$$

$$J_T(M^2)=\frac{1}{24\pi^2}\int_{4m_\pi^2}^{\infty}dsse^{-s/M^2}v^3n(\sqrt{s}/2), \tag{5.27}$$

with $v\equiv v(s)=\sqrt{1-4m_\pi^2/s}$. Observe that the sum rules (5.25-26) are not independent, in agreement with the relation (5.5) : the second one is obtained by differentiating the first with respect to $1/M^2$.

6. Evaluation of sum rules

In the above sum rules we have approximated the absorptive parts of the amplitudes by the ρ-meson pole and the 2π continuum, while we retain only the contributions of the unit operator (the perturbative result) and of all the operators of dimension four in the operator product expansion. To check this saturation scheme, let us compare the zero temperature limit of our sum rules with the corresponding vacuum sum rules [6]. The latter include, in addition, the rather large contribution from the high energy continuum beyond 1.5 GeV, as indicated by the experimental data, as well as the contribution of a quark operator of dimension six, which is also large because of its origin in Born (rather than loop) diagram. Since we do not include any of these contributions, we cannot expect the sum rules as written above to be well saturated.

Rather than incorporate these contributions, we isolate the thermal effects by considering the difference sum rules, obtained by subtracting out the vacuum sum rules from the corresponding finite temperature ones. Then the contribution to the absorptive parts beyond 1.5 GeV, being temperature independent, cancels out in the difference. Thus it is the temperature dependent contributions of the ρ-meson and of the 2π continuum, which should dominate the absorptive parts of these sum rules. Also the dimension six quark operator, O_6 say, contributes an amount $\sim\langle O_6\rangle-\langle 0|O_6|0\rangle$, which is insignificant in the immediate neighbourhood of $T=0$ and as the temperature rises, this contribution is overwhelmed by that of the two-gluon and other Lorentz non-invariant operators, as our estimates below for these operators show.

The difference sum rules for the two invariant amplitudes allow us to calculate the temperature dependence of the ρ-meson parameters,

$$\Delta m_\rho(T) \equiv m_\rho(T) - m_\rho$$
$$= \frac{m_\rho e^{m_\rho^2/M^2}}{2F_\rho^2}\left\{I_T - \frac{J_T}{m_\rho^2} - \left(\frac{1}{m_\rho^2} + \frac{1}{M^2}\right)\overline{\langle O\rangle}\right\}, \tag{6.1}$$

$$\Delta F_\rho(T) \equiv F_\rho(T) - F_\rho$$
$$= -\frac{e^{m_\rho^2/M^2}}{2F_\rho}\left\{\frac{J_T}{M^2} + \left(1 - \frac{m_\rho^2}{M^2}\right)I_T + \frac{m_\rho^2}{M^4}\overline{\langle O\rangle}\right\}, \tag{6.2}$$

with

$$\overline{\langle O\rangle} = \hat{m}\,\overline{\langle \bar{q}q\rangle} + \frac{\overline{\langle G^2\rangle}}{24} + \frac{4}{16+3n_f}\left\{\langle\Theta\rangle + a(M^2)\langle 16\Theta/3 - \Theta\rangle\right\}. \tag{6.3}$$

where the bar on the operators indicates subtraction of their vacuum expectation values Here we insert the experimental values for m_ρ and F_ρ, as these are well reproduced by the vacuum sum rules.

We now collect information on the operator contributions. The vacuum expectation value of the chiral condensate $\langle 0|\bar{q}q|0\rangle$ is known from the PCAC relation of Gell-Mann, Oakes and Renner.

$$2\hat{m}\langle 0|\bar{q}q|0\rangle = -F_\pi^2 m_\pi^2,$$

where $\hat{m} = \frac{1}{2}(m_u + m_d)$ and the pion decay constant F_π, defined by,

$$\langle 0|A_\mu^a|\pi^b(q)\rangle = -iq_\mu\delta^{ab}F_\pi,$$

has the value $F_\pi = 93.3$ MeV. Taking $\hat{m} = 7$ MeV [26], we get $\langle 0|\bar{q}q|0\rangle = -(225\,MeV)^3$. The vacuum expectation value of the two-gluon operator, as determined from the QCD sum rules [6], is $\langle 0|G^2|0\rangle = (330\,MeV)^4$.

The operator G^2 is related to the trace of the energy momentum tensor $\Theta_{\mu\nu}$ by the trace anomaly [27]. Normalising it to zero vacuum expectation value and taking thermal average, it gives

$$\overline{\langle G^2\rangle} \equiv \langle G^2\rangle - \langle 0|G^2|0\rangle$$
$$= -\frac{8}{9}\left(\langle\Theta_\mu^\mu\rangle - \sum_q m_q\overline{\langle\bar{q}q\rangle}\right) \tag{6.4}$$

The trace at finite temperature is given by $\langle\Theta_\mu^\mu\rangle = \langle\Theta\rangle - 3P$ where $\langle\Theta\rangle$ is the energy density and P the pressure.

The temperature dependence of both $\langle\bar{q}q\rangle$ and $\langle\Theta\rangle$ have been calculated in chiral perturbation theory [5]. Corrections due to nonzero quark masses as well as the contributions of the massive states (K, η, ρ, ...) have also been incorporated. However, as the authors point out, the validity of the perturbation theory and the use of dilute gas approximation to calculate the contribution of the massive states restrict these results to within a temperature of about 150 MeV. Thus the critical temperature T_c is, strictly speaking, beyond the range of validity of their calculation. Since, however, the order parameter $\langle\bar{q}q\rangle$ falls rapidly at the upper end of this range, one has only to extrapolate it a little further to get $T_c = 190\,MeV$.

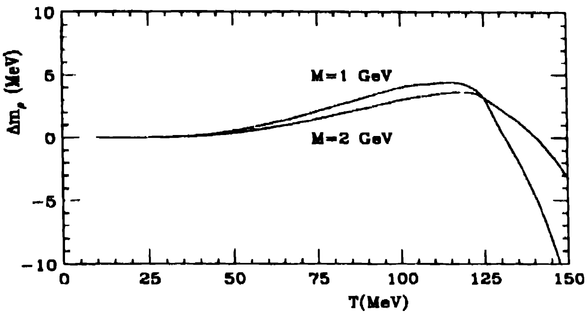

Figure 1 Shift in the rho-meson mass as a function of temperature for $M^2 = 1$ GeV^2 and $M^2 = 4$ GeV^2

Besides the total energy density $\langle\Theta\rangle$, there also occurs the thermal average, $\langle 16\,\Theta^f/3 - \Theta^g\rangle$. The last one cannot be calculated without further input, at least in the hadronic phase. Now both naive counting of the degrees of freedom and empirical study of the pion structure function [10] suggest the quark fraction of the energy density to be about half of the total. So we assume $\langle\Theta^f\rangle = \chi_f\langle\Theta\rangle$, with $\chi_f = .5$, whence $\langle 16\,\Theta^f - \Theta^g\rangle = \{(\frac{16}{3n_f}+1)\chi_f - 1\}\langle\Theta\rangle$.

As with the zero temperature sum rules, the results are expected to be stable in a region in M^2, which is neither too high to make the continuum contribution large relative to the resonance contribution nor too low to emphasize the neglected power corrections of higher order. Since the high energy continuum contribution gets cancelled in the difference sum rules, the region of M^2 may be extended somewhat at the upper end. Figs. 1 and 2 show the evaluation for M^2 equal to $1\,GeV^2$ and $4\,GeV^2$. The results for Δm_ρ and ΔF_ρ are rather stable for temperatures

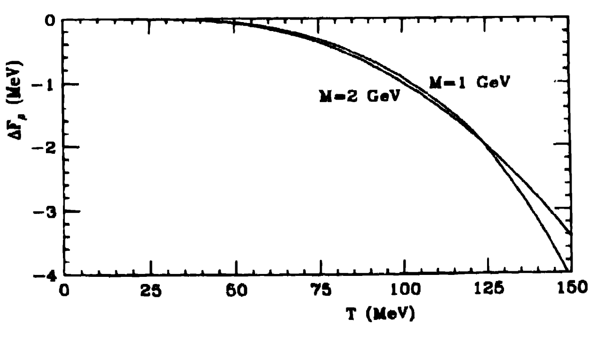

Figure 2 : Shift in the coupling of rho-meson with the vector current as a function of temperature for $M^2 = 1$ GeV^2 and $M^2 = 4$ GeV^2.

up to about 125 MeV. At higher temperatures the results, particularly for Δm_ρ appear unstable. Closer observation reveals here a large cancellation between the 2π contribution and the leading power correction we have retained. Thus the non-leading power correction become important here, whose inclusion may restore the stability in M^2 to higher temperatures.

7. Discussions

In this work we have written the thermal QCD sum rules for the two point correlation function of vector currents in the ρ-channel, including the leading power correction due to all the operators of dimension four. Because of the loss of Lorentz invariance at finite temperature, two new (Lorentz noninvariant) operators creep up in the Wilson expansion, in addition to the two old (Lorentz invariant) ones, already existing in the vacuum sum rules. Thus compared to the two numbers, $\langle 0|\bar{q}q|0\rangle$ and $\langle 0|G^2|0\rangle$ in the vacuum sum rules, we have now four temperature dependent quantities, $\langle\bar{q}q\rangle$, $\langle G^2\rangle$, $\langle\Theta\rangle$ and $\langle 16\,\Theta^f/3-\Theta^g\rangle$ in the thermal sum rules. All these thermal averages can be evaluated from chiral perturbation theory, supplemented by contributions from the massive states. An extra input is needed only for the last unphysical operator. The sum rules can then be used to determine the temperature dependence of the ρ-meson parameters. Our evaluation shows that the mass of the ρ-meson and its coupling with the vector current remain practically unaffected by the rise of temperature up to at least 125 MeV. The absence of the shift in the mass appears to agree with one set of results obtained in ref [28].

Unfortunately the sum rules, as they stand, cannot be extended up to the critical temperature. One reason contributing to this restriction has to do with the evaluation of the thermal average of the operators, as we already discussed in the last section. What restricts it further is their instability under a change of M^2 for temperatures above $125 MeV$. Even in this restricted temperature range, the numerical evaluation shows that the new operators are significant. In fact, for temperatures above say 70 MeV, the new contributions overwhelm the old ones in the difference sum rules. This situation necessitates reanalysis of all earlier results based on thermal QCD sum rules, where the new operators are not properly taken into account

We now discuss a possible way to extend the sum rules to higher temperatures. In the vacuum sum rules the operators of dimension four (and higher) provide corrections to the leading (perturbative) result. But in the difference sum rules it is these corrections which become the leading contribution. One thus expects that by including nonleading contributions from higher dimension operators along with those of dimension four already included, the sum rules would be stable against variations in M^2 over a wider range of temperature. This inclusion is all the more necessary for sum rules like the one for Δm_ρ, where the leading operator contribution cancels largely with that of $\pi\pi$-continuum.

The higher dimension operators will, of course, complicate the evaluation of the sum rules in that we have to know the temperature dependence of their thermal averages. Also there is more proliferation of operators than what is generally thought. The procedure in the literature [9] of including dimension six quark operators and excluding the Lorentz noninvariant gluon operators [29], because of the smallness of their coefficients by a factor of $\alpha_s(M^2)$, is not justified. For, as we have seen, the quark and the gluon operators mix under a change of scale, so that after renormalisation group improvement, both the coefficients are of the same order in $\alpha_s(M^2)$.

A way to proceed here is to write the entire set of sum rules by considering two-point functions of not only the vector quark bilinear but also the others, like the scalar, tensor, *etc.* All of these sum rules receive contribution from a few resonances and the operators from the same set. Further, using quark bilinears of appropriate chiralities, one can get sum rules without any of the gluon operators [30]. These sum rules should prove easier to evaluate and would also check the saturation scheme in a simpler context. The simultaneous consideration of all these sum rules is expected to provide a self-consistent evaluation of the thermal average of these operators. The sum rules will then pave the way to a fuller determination of the condensed matter characteristics of the QCD medium.

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References

[1] Y Nambu *Phys. Rev. Lett* **4** 380, 1960

[2] H Leutwyler *Non-relativistic effective Lagrangians*, preprint, BUTP93/25 and the references cited therein

[3] S Weinberg *The Quantum Theory of Fields* **vol.II** p.332 (Cambridge University Press, 1996)

[4] J Gasser and H Leutwyler *Ann. Phys* **158** 142 (1984); *Nucl. Phys* **B250** 465 (1985)

[5] P Gerber and H Leutwyler *Nucl. Phys* **B321** 387 (1989) See also T Gasser and H Leutwyler *Phys Lett* **184** 83 (1987) .

[6] M A Shifman, A I Vainshtein and V I Zakharov *Nucl. Phys.* **B147** 385 (1979) For a collection of original papers and comments, see *Vacuum Structure and QCD Sum Rules*, edited by M.A. Shifman (North Holland, Amsterdam, 1992). See also S. Narison, *QCD Spectral Sum Rules* (World Scientific, Singapore, 1989)

[7] K G Wilson *Phys. Rev.* **179** 171 (1969)

[8] A I Bochkarev and M E Shaposhnikov *Nucl Phys* **B268** 220 (1986)

[9] T Hatsuda, Y Koike and S H Lee *Nucl Phys.* **B394** 221 (1993). The earlier papers may be traced from this one.

[10] E V Shuryak *Rev. Mod. Phys* **65** 1, (1993)

[11] H A Weldon *Phys. Rev.* **D26** 1394 (1982)

[12] A J Niemi and G W Semenoff *Ann. Phys* **152** 105 (1984). See also N P Landsman and Ch G van Weert *Phys. Rep.* **145** 141 (1987)

[13] A V Smilga *Sov. J. Nucl Phys* **35** 271 (1982); V A Novikov, M A Shifman, A I Vainshtein and V I Zakharov *Forts d. Phys.* **32** 600 (1984)

[14] H Fritzsch and H Leutwyler *Phys. Rev.* **D10** 1624 (1974)

[15] W Hubschmid and S Mallik *Nucl. Phys.* **B207** 29 (1982)

[16] J Schwinger *Particles, Sources and Fields*, **volume I**, p. 271 (Addison-Wesley, Reading, 1970) ; C Cronstrom *Phys. Lett.* **B. 90**, 267, 1980

[17] R Courant and D Hilbert *Methoden der Mathematischen Physik II* (Springer- Verlag, 1968) p.434

[18] S Mallik *Phys. Lett.* **B 416** 373 (1998)

[19] L D Landau *Sov. Phys. -JETP* **7** 182 (1958), reproduced in *Collected Works of L D Landau*, edited by D Ter Haar (Pergamon Press, 1965)

[20] S Mallik, preprint (SINP) 1998

[21] J I Kapusta and E V Shuryak *Phys Rev.* **D49** 4694 (1994)

[22] H D Politzer *Phys. Rev. Lett.* **30** 1346 (1973) ; D J Gross and F Wilczek *Phys. Rev.* **D9** 980 (1974)

[23] See, for example, M E Peskin and DV Schroeder *An Introduction to Quantum Field theory* (Addison-Wesley, 1995)

[24] Particle Data Group *Phys. Rev.* **D54** 1 (1996)

[25] R L Kobes and G W Semenoff *Nucl. Phys.* **B260** 714 (1984)

[26] H Leutwyler *Phys. Lett.* **B 378** 313 (1996)

[27] H Leutwyler *Phys. Lett.* **B284** 106 (1992)

[28] C Song *Phys. Rev.* **D48**, 1375 (1993)

[29] The Lorentz invariant gluon operators of dimension six do not contribute to any of the correlation functions of two quark bilinears. See Ref.[15]

[30] S Mallik *Nucl Phys.* **B206** 90 (1982)